ARULMIGU PALANIANDAVAR ARTS COLLEGE FOR WOMEN, PALANI

Department of Mathematics

Learning Resources

Title of the paper: Complex Analysis

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Complex Analysis. $z = x + iy$ Complex Variable

<u>Unit I</u> Function of Complex Variable $w = f(z) = u(x,y) + i\,v(x,y)$

(i) $w = z^3$

$w = (x+iy)^3 = x^3 + 3x^2 iy + 3x(i^2y^2) + i^3y^3$

$= x^3 + i\,3x^2y - 3xy^2 - i\,y^3$

$w = x^3 - 3xy^2 + i(3x^2y - y^3)$.

$u(x,y) = x^3 - 3xy^2$, $v(x,y) = 3x^2y - y^3$

(ii) $w = 2\bar{z}^2 + 1$

$w = 2(\overline{x+iy})^2 + 1$

$= 2(x-iy)^2 + 1$

$= 2[x^2 - 2ixy + (i^2y^2)] + 1$

$= 2[x^2 - 2ixy - y^2] + 1$

$= 2x^2 - 4ixy - 2y^2 + 1$

$= (2x^2 - 2y^2 + 1) + i(-4xy)$

(iii) $w = \dfrac{1}{z}$

$= \dfrac{1}{x+iy}$

$= \dfrac{1}{x+iy} \times \dfrac{x-iy}{x-iy}$

$= \dfrac{x-iy}{x^2+y^2}$

$= \dfrac{x}{x^2+y^2} + i\left(\dfrac{-y}{x^2+y^2}\right)$

(iv) $w = \dfrac{z}{1+z}$

$= \dfrac{x+iy}{1+x+iy} = \dfrac{(x+iy)(1+x-iy)}{(1+x+iy)(1+x-iy)}$

$= \dfrac{x + x^2 - \cancel{ixy} + iy + \cancel{ixy} + y^2}{(1+x)^2+y^2}$

$= \dfrac{x^2+x+y^2}{(1+x)^2+y^2} + i\,\dfrac{y}{(1+x)^2+y^2}$

(v) $w = z\bar{z}$

$= (x+iy)(x-iy)$

$= x^2 + y^2$

(v) $w = z + \dfrac{1}{z} \Rightarrow w = x+iy + \dfrac{1}{x+iy} = \dfrac{(x+iy)^2+1}{x+iy}$

$w = \left(\dfrac{(x^2+2ixy+i^2y^2)+1}{(x+iy)}\right) \times \left(\dfrac{x-iy}{x-iy}\right)$

$= \dfrac{(x^2-y^2+1+2ixy)(x-iy)}{x^2+y^2} = \dfrac{x^3 - ix^2y - xy^2 + iy^3 + x - iy + 2ix^2y + 2xy^2}{x^2+y^2}$

$= \dfrac{x^3+x+xy^2+i(x^2y-y+y^3)}{x^2+y^2} = \dfrac{x^3+x+xy^2}{x^2+y^2} + i\left(\dfrac{x^2y-y+y^3}{x^2+y^2}\right)$

Elementary Transformation.

1] Translation. $w = z + b$, $z = x + iy$, $b = b_1 + ib_2$.

$$w = x + iy + b_1 + ib_2$$

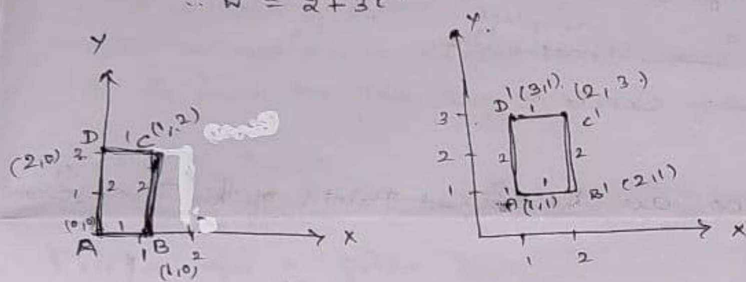
$$w = (x + b_1) + i(y + b_2).$$

★ The image of the point (x, y) in the z-plane is the point $(x+b_1, y+b_2)$ in the plane.

Take $z = 1 + 2i$ $\quad b = 1 + i$.

$w = z + b = 1 + 2i + 1 + i$

$\therefore w = 2 + 3i$



z-plane $\qquad$ w-plane.

Two regions have Same shape, size & Orientation.

★ Straight line → straight line

★ Circle → Circle → with centre a radius r. ⇒ centre a+b and radius r.

Z-plane $\qquad$ w-plane.

$w = z + b \Rightarrow z = z + b$. z is not define.

Translation. → [∞ is a fixed Point].

Rotation. $w = az$, $|a| = 1$.

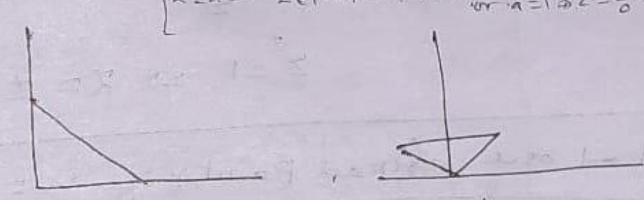
[$z = az$, $z(1-a) = 0$. either $z = 0$. or $a = 1 \Rightarrow z = \frac{0}{0} = \infty$]

$z = re^{i\theta}$, $a = e^{i\alpha}$

$|a| = 1$.

$w = az = re^{i\theta} \times e^{i\alpha}$

$w = re^{i(\theta+\alpha)}$.



z-plane $\qquad$ w-plane.

in z-plane (r, θ)

⇒ w plane $(r, \theta + \alpha)$.

Straight line → Straight line

circle → circle.

Rotation.

(0 and ∞ are the two fixed Point)

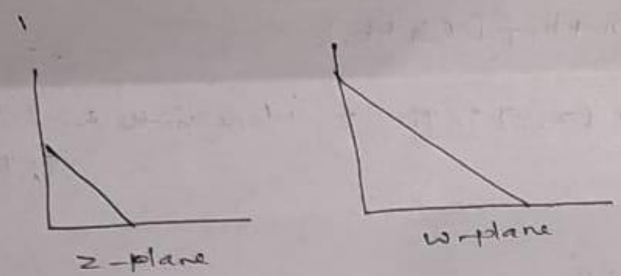
Magnification or Contraction.

$W = bz$ ($b>0$, real).

$W = bz$, b is real & $b>0$.

in z plane polar coordinates (r, θ).

∴ W plane " (br, θ).



Magnification.

Mapping z-plane → w-plane,

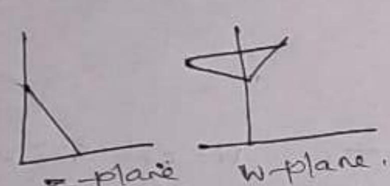
Straight line → Straight line

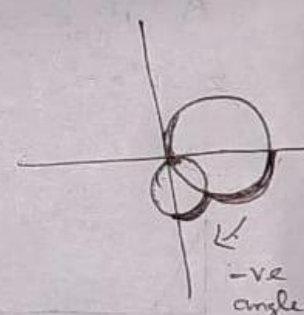
Circle → circle.

0 and ∞ are the fixed points of this Transformation

Inversion : $W = \frac{1}{z}$. , put $z = re^{i\theta}$

$W = (1/r)\, e^{-i\theta}$.





$W = az + b$, a and b are complex numbers.

Find

Fixed Point $W = \frac{1}{z}$ ⇒ Put $w = z$. we get

$z = \frac{1}{z}$

$z^2 = 1$ ⇒ $z = \pm 1$.

$\boxed{\therefore \; 1 \text{ and } -1 \text{ are fixed Points of } W = \frac{1}{z}}$

$w = \frac{1}{z}$

This transformation can be expressed as a product of two transformations.

$T_1(z) = \left(\frac{1}{r}\right) e^{i\theta}$, $T_2(z) = re^{-i\theta} = \bar{z}$.

(ie) $(T_1 \circ T_2)(z) = T_1(T_2(z)) = T_1(re^{i\theta})$
$= \frac{1}{r}(e^{-i\theta}) = \frac{1}{z}$

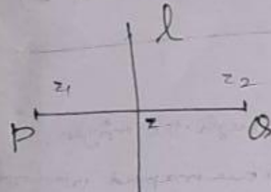
Defn

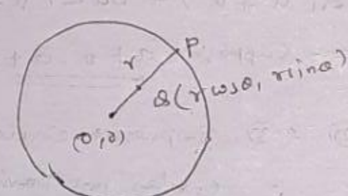
Inversion.

Two points P and Q are said to be inverse points with respect to a circle with centre O and radius r if Q lies on the ray OP and $OP \cdot OQ = r^2$.

Reflection Points.

Two points P and Q are called reflection points for a given straight line l iff l is the perpendicular bisector of the segment PQ.





The transformation $T_1(z) = \left(\frac{1}{r}\right)e^{i\theta}$ represents the inversion with respect to the unit circle $|z| = 1$ and $T_2(z) = \bar{z}$ represents reflection about the real axis.

Hence $w = \frac{1}{z}$ is the inversion w.r.t the unit circle followed by the reflection about the real axis.

$OP = r$

$OQ = \sqrt{(0 - r\cos\theta)^2 + (0 - r\sin\theta)^2}$
$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$
$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$
$= \sqrt{r^2 \times 1} = r$

$\therefore OP \cdot OQ = r \times r = r^2$

~~PQ are~~

Thus P and Q are inverse points.

(8)

$w = u + iv = \frac{1}{z}$

$w = u + iv = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$

$w = \frac{x-iy}{x^2+y^2} \Rightarrow w = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$

$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$

III^{ly} $z = \frac{1}{w} \Rightarrow z = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$. —— (1)

Now consider the equation.

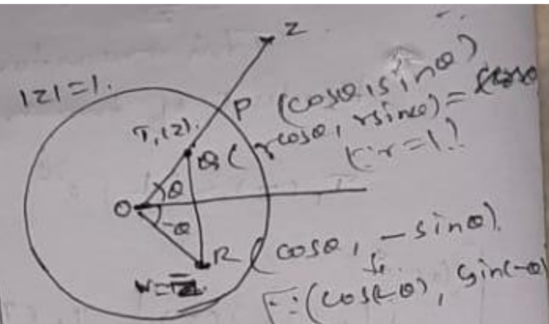
$a(x^2+y^2) + bx + cy + d = 0$. —— (2)

where a, b, c, d are real.

This equation represents circle or straight line according as $a \neq 0$ or $a = 0$.

using (1) in (2) we get

$d(u^2+v^2) + bu - cv + a = 0$ —— (3)



Q R reflection point.

Q lies on OP.

OP = 1.

OQ = 1.

OP · OQ = 1.

$[\because OP \cdot OQ = r^2]$

P and Q are inversion w.r.t unit circle $|z| = 1$.

Now suppose $a \neq 0, d \neq 0$.

(2) & (3) represent circles not passing through the origin.

∴ circles not passing through the origin are mapped into circles not passing through the origin.

$a = 0, d \neq 0$. (2) straight line. $bx + cy + d = 0$. not passing through the origin, mapped into.

(3) circle $d(u^2+v^2) + bu - cv = 0$. Passing through the origin.

III^{ly} $a \neq 0$ $d = 0$.

Circle passing through the origin mapped into straight line not passing through the origin.

* $a = 0, d = 0$

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Ex 1 $w = iz + i$

Let $z = x + iy$.

$w = i(x+iy) + i$

$u + iv = -y + i(x+1)$.

$u = -y$, $v = x+1$.

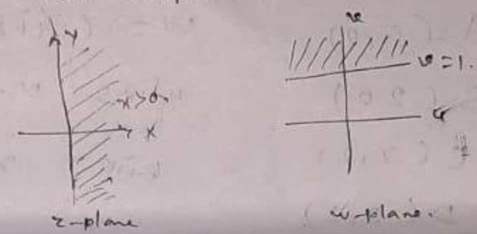
Given $x > 0$.

Then $v = x + 1$

$v > 1$. $\because x > 0$.

$x > 0 \iff v > 1$.

$x > 0$ maps onto $v > 1$.



Ex 2. $w = iz + 1$.

$z = x + iy$

then $w = i(x+iy) + 1$

$= ix - y + 1$

$u + iv = 1 - y + ix$.

$u = 1 - y$, $v = x$.

Given $x > 0$ and $0 < y < 2$.

$x > 0 \Rightarrow v = x$

$\Rightarrow v > 0$.

$0 < y < 2 \Rightarrow u = 1 - y$.

$y > 0 \Rightarrow -y < 0$

$\therefore u < 1$

$y < 2 \Rightarrow -y > -2$.

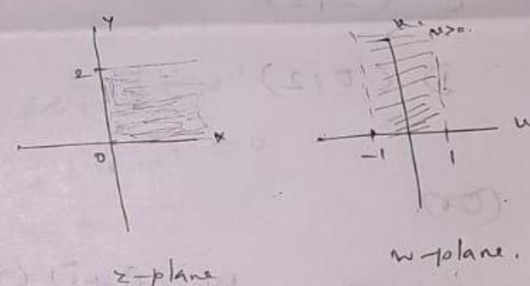
$u = 1 - y$.

$\therefore u > 1 - 2$ $(\because -y > -2)$.

$u > -1$

$\boxed{\therefore 0 < y < 2 \Rightarrow -1 < u < 1}$

In z-plane $x > 0$ & $0 < y < 2$ mapped into w plane $-1 < u < 1$ & $v > 0$.



$x > 0$ and $0 < y < 2$ mapped $v > 0$ and $-1 < u < 1$.

Ex 3 Find the image of the Square $(0,0)$ $(2,0)$ $(2,2)$ $(0,2)$ under the transformation $w=(1+i)z+(2+i)$.

$A=(0,0)$ $\quad w=(1+i)(x+iy)+(2+i)$

$B(2,0)$ $\quad u+iv = x+iy+ix-y+2+i$

$C(2,2)$ $\quad u+iv = x-y+2+i(x+y+1)$.

$D(0,2)$.

$$u = x-y+2, \quad v = x+y+1.$$

	z-plane (x,y)	w-plane (u,v)	
A	$(0,0)$	$(2,1)$	$\to A'$
B	$(2,0)$	$(4,3)$	$\to B'$
C	$(2,2)$	$(2,5)$	$\to C'$
D	$(0,2)$	$(0,3)$	$\to D'$

(or)

$w=(1+i)(x+iy)+(2+i)$

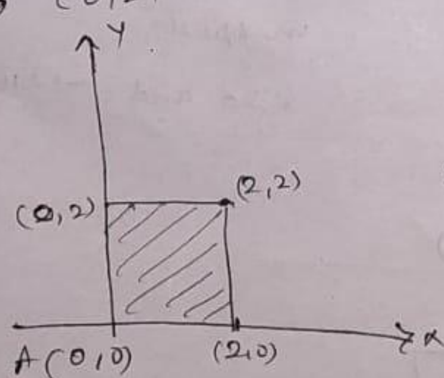
~~A(0,0)~~

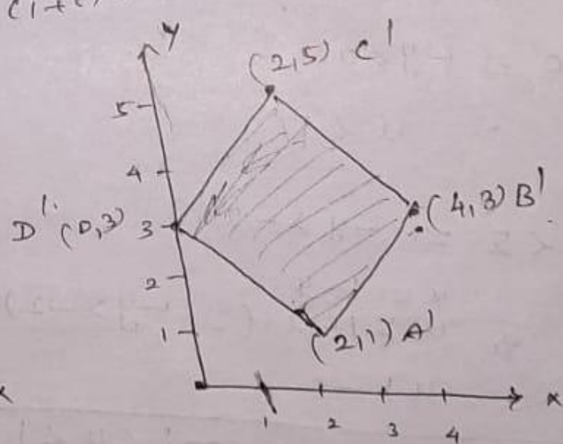
$A(0,0) \to A'=(1+i)(0+i0)+2+i = 2+i \to (2,1)$

$B(2,0) \to B'=(1+i)(2+i0)+2+i = 4+3i \to (4,3)$

$C(2,2) \to C'=(1+i)(2+2i)+2+i = 2+5i \to (2,5)$

$D(0,2) \to D'=(1+i)(0+2i)+2+i = 0+3i \to (0,3)$.





(0,2)

Formula.

$$x^2+y^2+2gx+2hy+f=0.$$

Centre $(-g,-h)$ radius $=\sqrt{g^2+h^2-f}$.

Ex4 $w=\frac{1}{z}$. Find in z-plane $|z-3|=5$. mapped into $|w+\frac{3}{16}|=\frac{5}{16}$.

The circle $|z-3|=5$.

$$\left|\frac{1}{w}-3\right|=5.$$

$$\left|\frac{1}{u+iv}-3\right|=5 \Rightarrow \left|\frac{1-3u-3iv}{u+iv}\right|=5 \Rightarrow |1-3u-3iv|=5|u+iv|$$

$$\therefore |1-3u+i(-3v)|=5|u+iv|.$$

$$\sqrt{(1-3u)^2+(-3v)^2}=5\sqrt{u^2+v^2}.$$

Take square on both sides we get

$$(1-3u)^2+(-3v)^2=5^2(u^2+v^2)$$

$$1+9u^2-6u+9v^2=25u^2+25v^2$$

$$25u^2+25v^2-9u^2-9v^2+6u-1=0.$$

$$16u^2+16v^2+6u-1=0.$$

2,1)

(4,3)

$\to$(2,5)

$\to$(0,3).

$$u^2+v^2+\frac{6}{16}u-\frac{1}{16}=0.$$

$$u^2+v^2+2\left(\frac{3}{16}\right)u+2\times0\times v+\left(-\frac{1}{16}\right)=0.$$

$$\Rightarrow u^2+v^2+2gu+2hv+f=0.$$

Take, $g=\frac{3}{16}$, $h=0$, $f=-\frac{1}{16}$.

This is the circle with Centre $\boxed{(-g,-h)}$ is $\left(-\frac{3}{16},0\right)$.

$$\text{radius}=\sqrt{g^2+h^2-f}=\sqrt{\left(\frac{3}{16}\right)^2+0-\left(-\frac{1}{16}\right)}=\sqrt{\frac{9}{16^2}+\frac{1}{16}}$$

$$=\sqrt{\frac{9+16}{(16)^2}}=\sqrt{\frac{25}{(16)^2}}=\frac{5}{16}.$$

$\therefore$ The image circle in the w plane is $\left|w+\frac{3}{16}\right|=\frac{5}{16}$.

Find the image of the circle $|z-3i|=3$.

Ex 5 $w = \frac{1}{z}$,

Soln $|z-3i| = 3$

$\left|\frac{1}{w} - 3i\right| = 3 \Rightarrow \left|\frac{1}{u+iv} - 3i\right| = 3 \Rightarrow |1 - 3iu + 3v| = 3|u+iv|$

$\Rightarrow |1+3v + i(-3u)| = 3|u+iv|$.

$\Rightarrow \sqrt{(1+3v)^2 + (-3u)^2} = 3\sqrt{u^2+v^2}$

$\Rightarrow (1+3v)^2 + (-3u)^2 = 3^2(u^2+v^2)$

$\Rightarrow 1 + 9v^2 + 6v + 9u^2 = 3^2(u^2+v^2)$

$\Rightarrow 1 + 9v^2 + 6v + 9u^2 = 9u^2 + 9v^2$

$\Rightarrow 1 + 6v = 0$.

$\Rightarrow 6v + 1 = 0$.

The image of the circle $|z-3i| = 3$ under $w = \frac{1}{z}$ in the z-plane is the straight line $6v+1=0$ in the w-plane.

Ex 6 $w = \frac{1}{z}$ Find the image of the strip $2 < x < 3$.

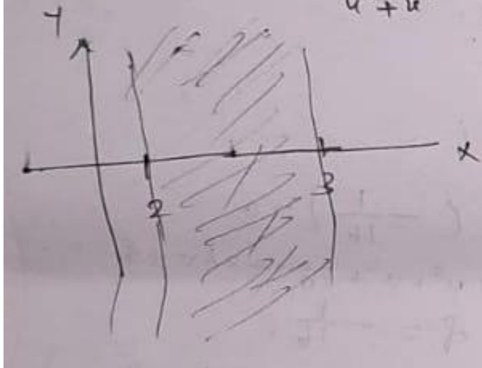
Soln $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$z = \frac{1}{u+iv} \Rightarrow z = \frac{u}{u^2+v^2} + \frac{(-iv)}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

$x \geq 2 \Rightarrow \frac{u}{u^2+v^2} > 2 \Rightarrow u > 2(u^2+v^2) \Rightarrow 2u^2 + 2v^2 - u < 0$.

$\Rightarrow u^2 + v^2 - \frac{u}{2} < 0$.



$u^2 + v^2 + 2gu + 2hv + f = 0$.

$g = $.

$u^2 + v^2 + 2\left(-\frac{1}{4}\right)u + 2 \times 0 \times v + 0 < 0$

$g = -\frac{1}{4}$, $h = 0$, $f = 0$

Centre $(-g, -h)$

$\left(\frac{1}{4}, 0\right)$

radius $\sqrt{g^2+h^2-f}$

$\sqrt{\left(-\frac{1}{4}\right)^2}$

$u^2 + v^2 - \frac{u}{2} < 0$.

$x > 2 \Rightarrow$ Interior of the Circle centre $\left(\frac{1}{4}, 0\right)$ radius $\frac{1}{4}$.

(15) $x < 3 \Rightarrow \frac{u}{u^2+v^2} < 3 \Rightarrow u < 3(u^2+v^2) \Rightarrow u < 3u^2+3v^2$

$\Rightarrow 3u^2+3v^2-u > 0.$

$\Rightarrow u^2+v^2-\frac{u}{3} > 0$

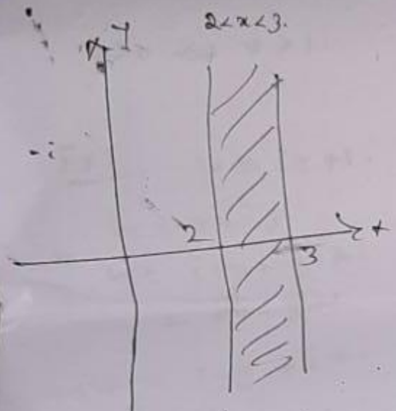
$\Rightarrow u^2+v^2+2\left(-\frac{1}{6}\right)u+2\times 0\times v+0 > 0.$

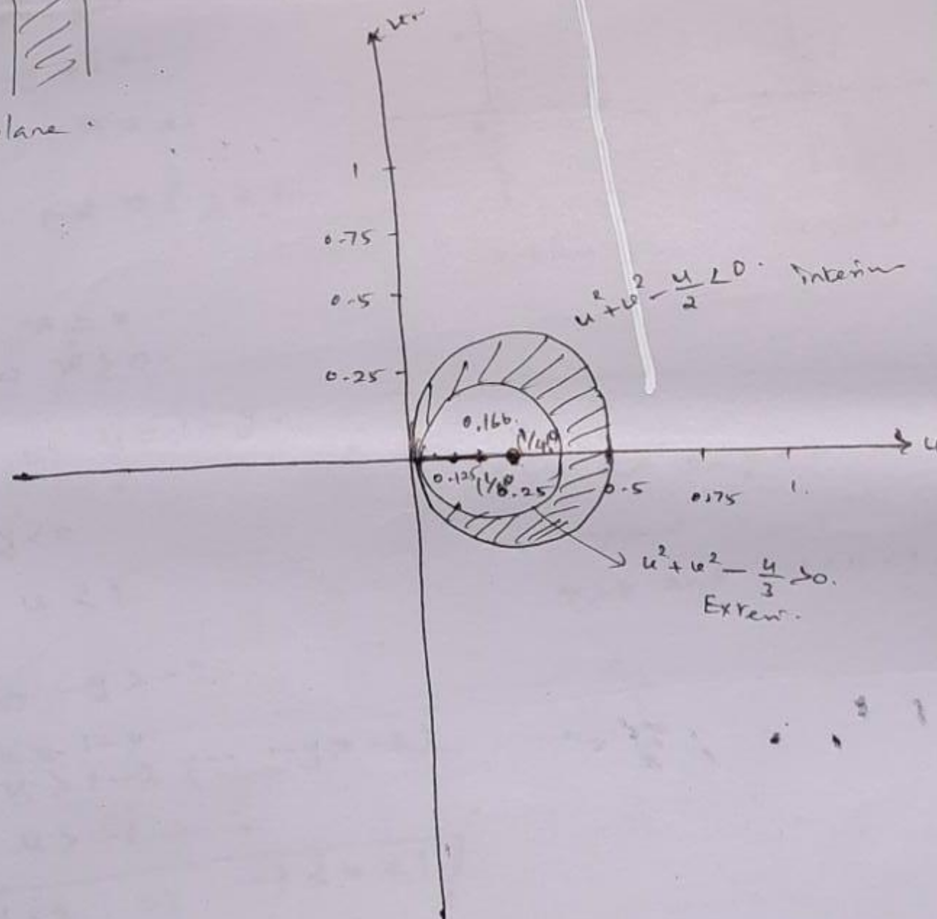
$g = -\frac{1}{6}, \quad h = 0 \quad f = 0$

centre $\left(-\left(-\frac{1}{6}\right), 0\right) = \left(\frac{1}{6}, 0\right)$

radius $\sqrt{\left(-\frac{1}{6}\right)^2+0^2+0} = \frac{1}{6} = 0.166.$

$x < 3 \Rightarrow u^2+v^2-\frac{u}{3} > 0 \Rightarrow$ Exterior of the circle with centre $\left(\frac{1}{6}, 0\right)$ & radius $\frac{1}{6}$.





Limits:

A function $w = f(z)$ is said to have the limit l as z tends to z_0. If given $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$

$0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$

we write $\lim_{z \to z_0} f(z) = l$.

Lemma When the limit of a function $f(z)$ exists as z tends to z_0 then the limit has a unique value.

Proof Suppose that $\lim_{z \to z_0} f(z)$ has two values l_1 and l_2

Then given $\epsilon > 0$, There exists δ_1 and $\delta_2 > 0$ such that

$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - l_1| < \epsilon/2$

$0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - l_2| < \epsilon/2$

Now let $\delta = \min\{\delta_1, \delta_2\}$.

Then if $0 < |z - z_0| < \delta$ we have

$$|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2|.$$

$$\leq |f(z) - l_1| + |f(z) - l_2|.$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon \quad \left[\because |x+y| \leq |x| + |y|. \text{ Triangle inequalities}\right]$$

Since $\epsilon > 0$ is arbitrary $|l_1 - l_2| = 0$ so that $l_1 = l_2$.

Ex 1 Let $f(z) = \begin{cases} z^2 & \text{if } z \neq i \\ 0 & \text{if } z = i \end{cases}$

Prove that $\lim_{z \to i} f(z) = -1$.

Soln T.P.T $\lim_{z \to i} f(z) = -1$.

(i.e) To prove that $0 < |z - i| < \delta \Rightarrow |z^2 + 1| < \epsilon$.

$\because |z^2 + 1| = |(z+i)(z-i)|$.

$|z^2 + 1| = |z+i||z-i|$. —①

If we can find $\delta > 0$ satisfying the requirements of the definition then we can choose another $\delta \le 1$. satisfying the requirements of the definition

now $0 < |z-i| < 1 \Rightarrow |z+i| = |z-i+2i|$.

$$\le |z-i| + |2i|.$$

$$< 1+2 = 3.$$

$\therefore |z+i| < 3$.

using eqn ① we obtain:

$$0 < |z-i| < 1 \Rightarrow |z^2+1| < 3|z-i| < 3\delta.$$

$$< \epsilon/3$$

Hence if we choose $\delta = \min\left\{1, \frac{\epsilon}{3}\right\}$ we get

$$0 < |z-i| < \delta \Rightarrow |z^2+1| < 3 \times \epsilon/3 = \epsilon.$$

$$\therefore 0 < |z-i| < \delta \Rightarrow |z^2+1| < \epsilon.$$

$$\therefore \lim_{z \to i} f(z) = -1.$$

Ex 2 Show that $\displaystyle\lim_{z \to 2} \frac{z^2-4}{z-2} = 4$.

Soln Let $f(z) = \dfrac{z^2-4}{z-2}$. $\qquad f(2) = \dfrac{0}{0}$.

Hence $f(z)$ is not defined at $z = 2$.

and when $z \ne 2$ we have.

$$f(z) = \frac{(z+2)(z-2)}{(z-2)} = z+2.$$

$\therefore |f(z) - 4| = |z+2-4| = |z-2|$ when $z \ne 2$

now given $\epsilon > 0$, we choose $\delta = \epsilon$.

Then $0 < |z-2| < \delta \Rightarrow |f(z) - 4| < \epsilon$.

$$\therefore \lim_{z \to 2} f(z) = 4.$$

Ex 3 The function $f(z) = \dfrac{\bar{z}}{z}$ does not have a limit as $z \to 0$.

Soln

$$f(z) = \frac{\bar{z}}{z} = \frac{x - iy}{x + iy}.$$

Suppose $z \to 0$ along the $y = mx$.

Along this path $f(z) = \dfrac{x - imx}{x + imx} = \dfrac{1 - im}{1 + im}$ as $x \neq 0$.

Hence if $z \to 0$ along the path $y = mx$.

$f(z)$ tends to $\dfrac{1 - im}{1 + im}$ which is different for different values of m.

Hence $f(z)$ does not have a limit as $z \to 0$.

Exercises Page 28.

P.T the function $f(z)$ does not have a limit as $z \to 0$

(i) $f(z) = \dfrac{\sqrt{|xy|}}{x + iy}$, $z = x + iy \neq 0$.

(ii) $f(z) = \dfrac{-ix^3 y}{x^6 + y^2}$, $z \neq 0$, (iii) $f(z) = \dfrac{xy}{x^2 + y^2}$, $z \neq 0$.

Soln

(i). $f(z) = \dfrac{\sqrt{|xy|}}{x + iy}$.

Suppose $z \to 0$ along the $y = mx$.

Along this path $f(z) = \dfrac{\sqrt{|x\,mx|}}{x + imx} = \dfrac{\sqrt{|x^2 m|}}{x(1 + im)} = \dfrac{x\sqrt{|m|}}{x(1 + im)}$.

$$f(z) = \frac{\sqrt{|m|}}{1 + im}.$$

$f(z)$ tends to $\dfrac{\sqrt{|m|}}{1 + im}$ which is different for different value of m.

Hence $f(z)$ does not have a limit as $z \to 0$.

(ii) $f(z) = \frac{-ix^3y}{x^6+y^2}$.

Along the path $y = mx^3$. we have $f(z) = \frac{-ix^3\, mx^3}{x^6+m^2x^6}$

$$f(z) = \frac{-ix^6 m}{x^6(1+m^2)}$$

$$f(z) = \frac{-im}{1+m^2}$$

Hence if $z \to 0$ along the path $y = mx^3$.

$f(z)$ tends to $\frac{-im}{1+m^2}$ which depends on m.

Hence $f(z)$ does not have a limit as $z \to 0$.

Sln. (iii) $f(z) = \frac{xy}{x^2+y^2}$, $z \neq 0$.

Soln Along the path $y = mx$ we have $f(z) = \frac{x\, mx}{x^2+m^2x^2}$

$$f(z) = \frac{m}{1+m^2}.$$

Hence if $z \to 0$ along the path $y = mx$.

$f(z)$ tends to $\frac{m}{1+m^2}$ which depends on m.

Hence $f(z)$ does not have a limit as $z \to 0$.

Ex 4 $f(z) = \frac{x^2y^2}{(x+y^2)^2}$, $z \neq 0$

Along the parabola $y^2 = mx$. we have $f(z) = \frac{mx^3}{(x+mx)^2} = \frac{m}{(1+m)^2}$

Hence if $z \to 0$ along the parabola $y^2 = mx$.

$f(z)$ tends to $\frac{m}{(1+m)^2}$ which depends on m.

Hence $f(z)$ does not have a limit as $z \to 0$.

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